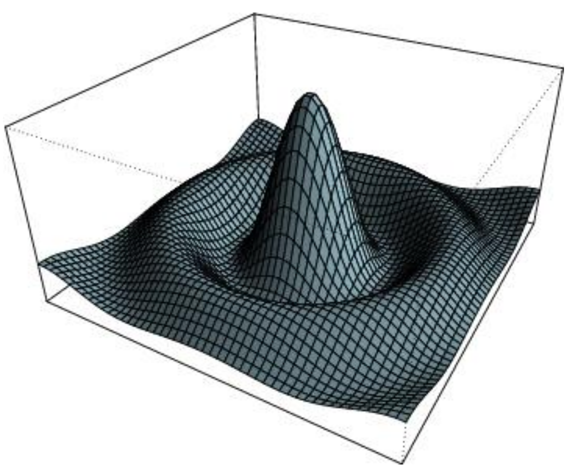
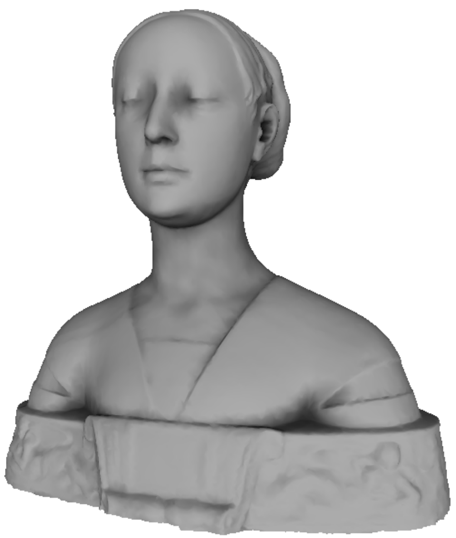
Data structures for 3D Meshes

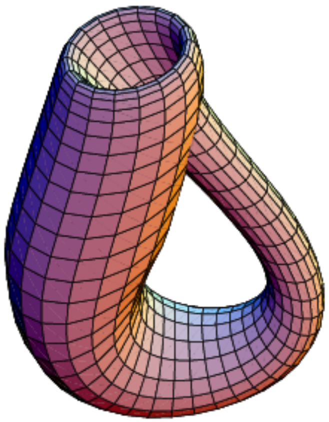
Paolo Cignoni
p.cignoni@isti.cnr.it
http://vcg.isti.cnr.it/~cignoni

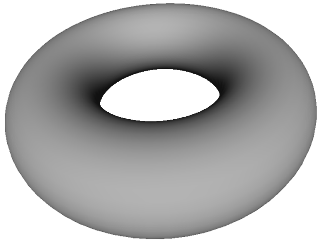
Surfaces

- A 2-dimensional region of 3D space
- *A portion of space having length and breadth but no thickness*







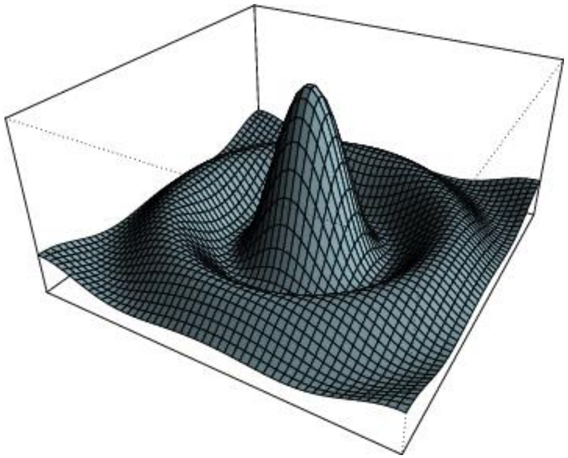


Defining Surfaces

- Analytically…
  - Parametric surfaces

$$S: R^2 \rightarrow R^3$$

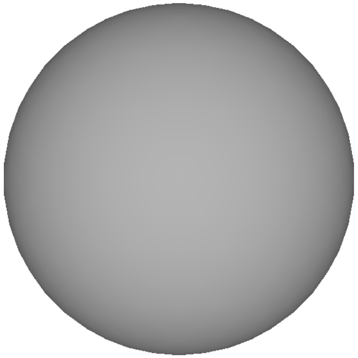
$$S(x,y) = \left(x, y, \sin\left(\sqrt{(x^2+y^2)}\right)/\sqrt{(x^2+y^2)}\right)$$



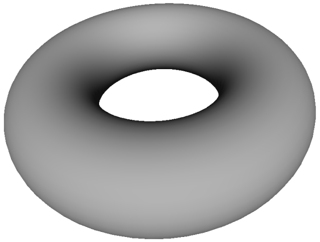
  - Implicit surfaces

$$S = \{(x,y,z): f(x,y,z) = 0\}$$

$$S = \{(x,y,z): x^2 + y^2 + z^2 - r^2 = 0\}$$



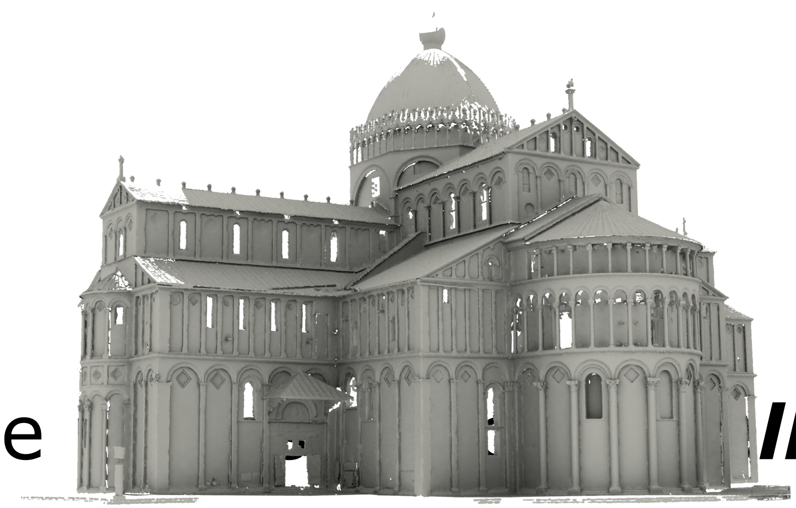
$$S = \{(x,y,z): (x^2 + y^2 + R^2 - r^2)^2 - 4R^2(x^2 + y^2) = 0\}$$



Representing Real World Surfaces

- Analytic definition falls short of representing *real world* surfaces in a *tractable* way

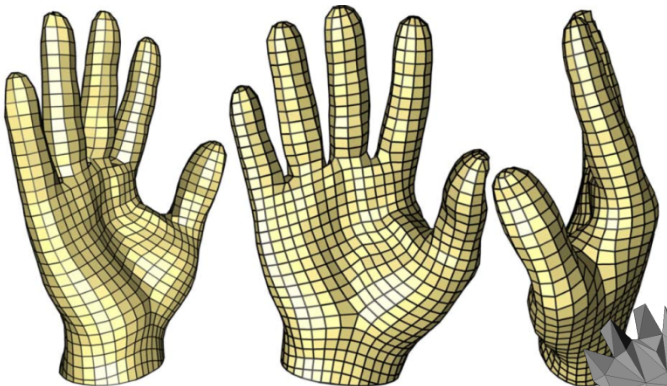
$$S(x,y) = \ldots ?$$



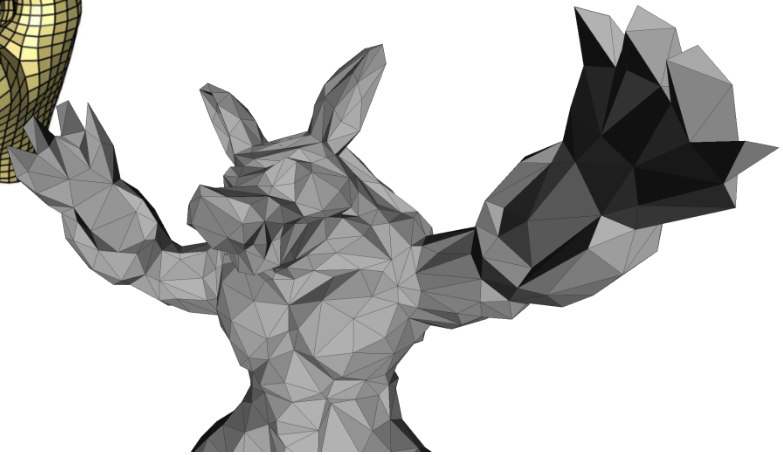
... surfaces can be [illegible] ***ll complexes***

Cell complexes (meshes)

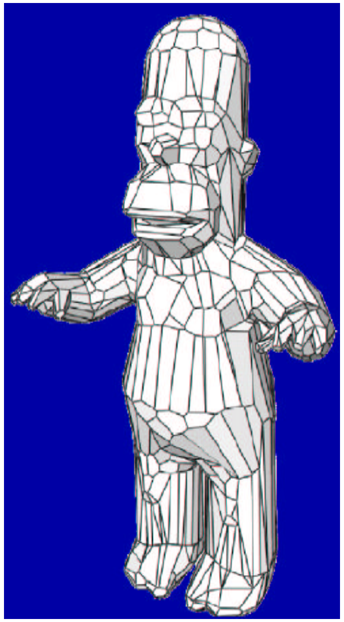
- Intuitive description: a continuous surface divided in polygons



quadrilaterals (quads)



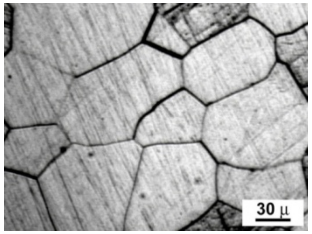
triangles

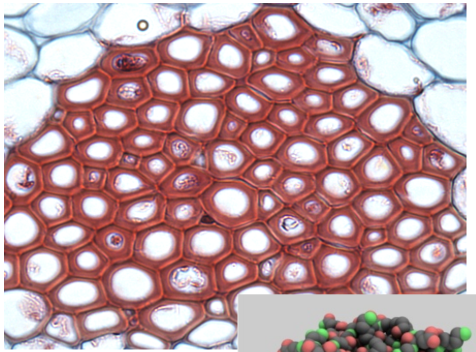
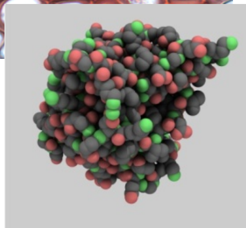


Generic polygons

Cell Complexes (meshes)

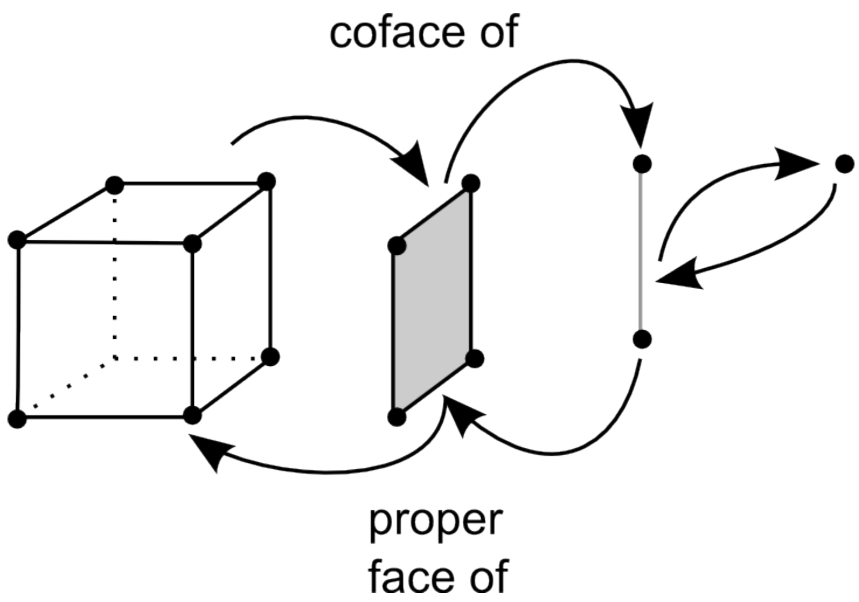
- In nature, meshes arise in a variety of contexts:
  - Cells in organic tissues
  - Crystals
  - Molecules
  - …
  - Mostly *convex* but *irregular* cells
  - Common concept: *complex* shapes can be described as *collections* of *simple building blocks*



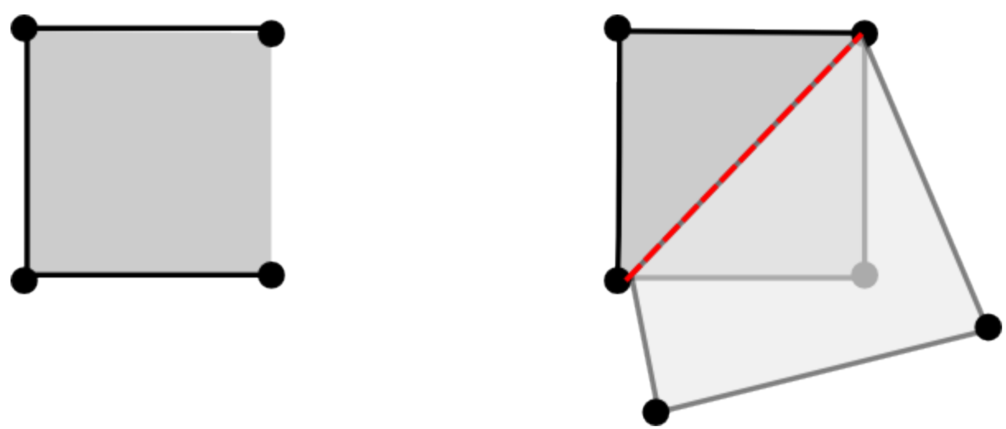
Cell Complexes (meshes)

- Slightly more formal definition
  - a *cell* is a convex polytope in
  - a *proper face* of a cell is a lower dimension convex polytope subset of a cell



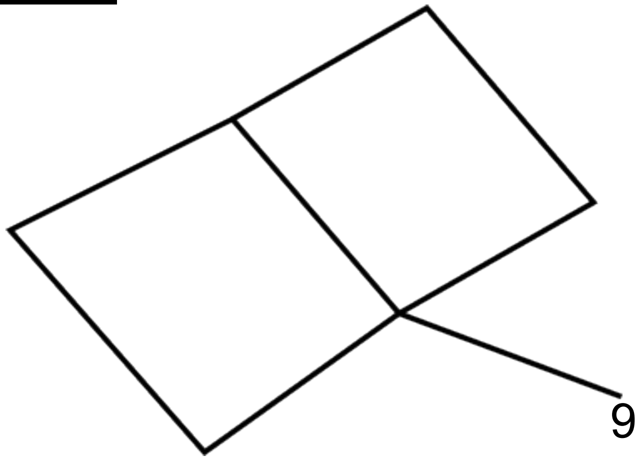
Cell Complexes (meshes)

- a collection of cells is a complex **iff**
  - every face of a cell belongs to the complex
  - For every cells C and C′, their intersection either is empty or is a common face of both



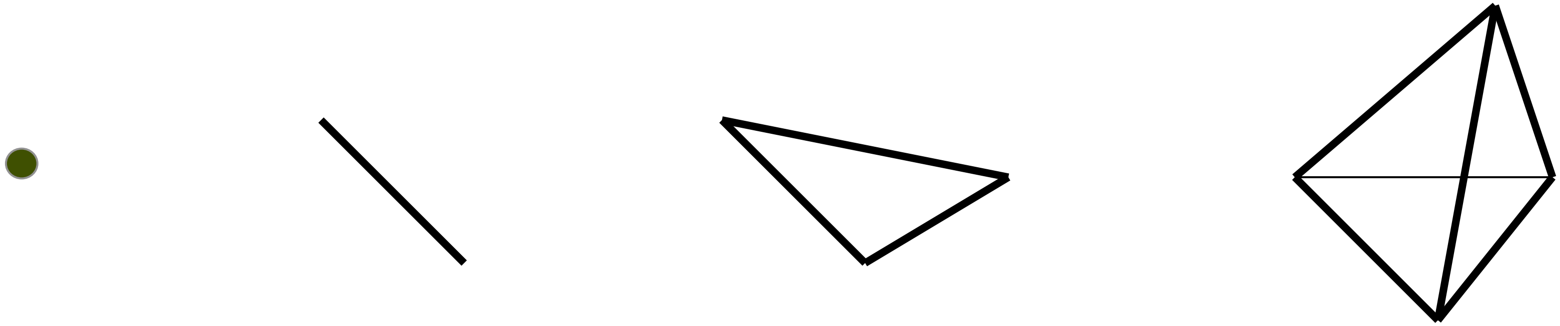
Maximal Cell Complex

- the **order** of a cell is the number of its sides (or vertices)
- a complex is a **k-complex** if the maximum of the order of its cells is *k*
- a cell is **maximal** if it is not a face of another cell
- a k-complex is **maximal** *iff* all maximal cells have order k
- short form : <u>no dangling edges</u>!



Simplicial Complex

- A cell complex is a **simplicial complex** when the cells are simplexes
- A **d-*simplex*** is the convex hull of $d+1$ points in



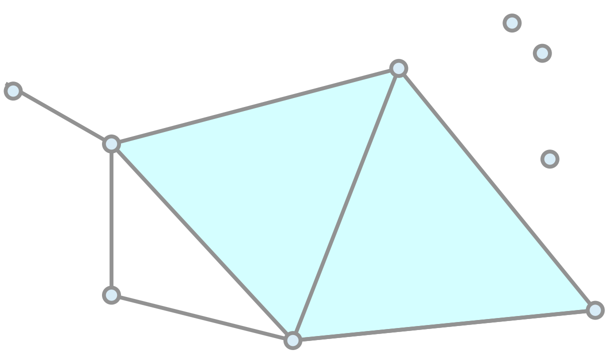
0-simplex 1-simplex 2-simplex 3-simplex

Sub-simplex / face

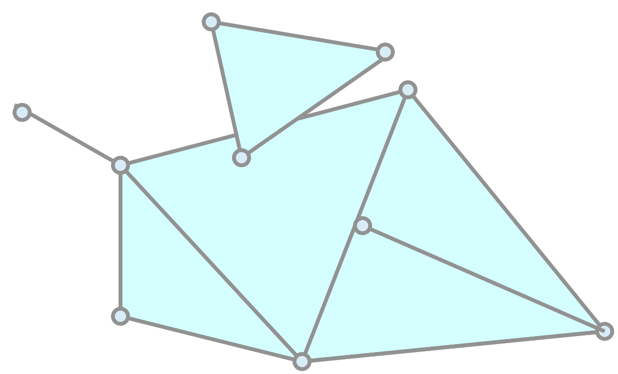
- A simplex σ' is called *face* of another simplex σ if it is defined by a subset of the vertices of σ
-
- If $\sigma \neq \sigma'$ it is a proper face

Simplicial Complex

- A collection of simplexes Σ is a simplicial k-complex iff:
  - $\forall\ \sigma_1, \sigma_2, \in \Sigma$
    $\sigma_1 \cap \sigma_2 \neq \emptyset \quad \Rightarrow \quad \sigma_1 \cap \sigma_2$ is a simplex of Σ
  - $\forall\ \sigma \in \Sigma$ all the faces of σ belong to Σ
  - k is the maximum degree of simplexes in Σ



OK

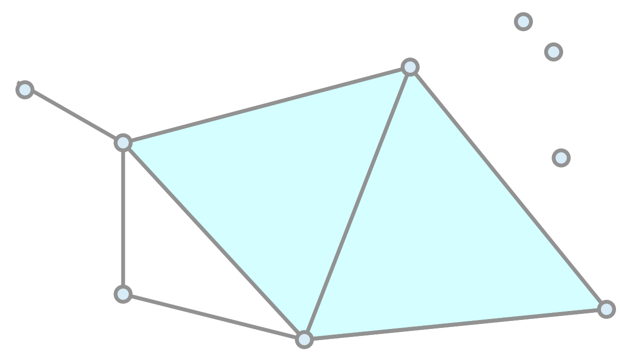


Not Ok

Simplicial Complex

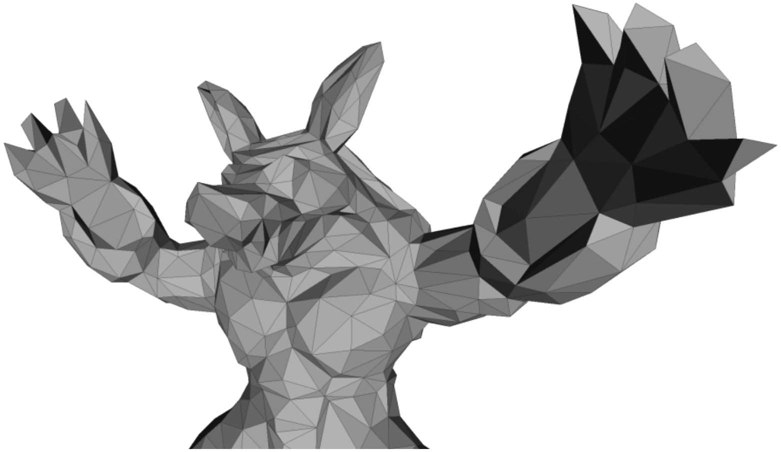
- A simplex σ is maximal in a simplicial complex Σ if it is not a proper face of a another simplex σ' of di Σ
- A simplicial k-complex Σ is maximal if all its maximal simplex are of order k
  - No dangling lower dimensional pieces

Non maximal 2-simplicial complex



Meshes, at last

- When talking of *triangle mesh* the intended meaning is a **maximal 2-simplicial complex**



Topology vs Geometry

- It is quite useful to discriminate between:
  - Geometric realization
    - **Where** the vertices are actually placed in space
  - Topological Characterization
    - **How** the elements are combinatorially connected

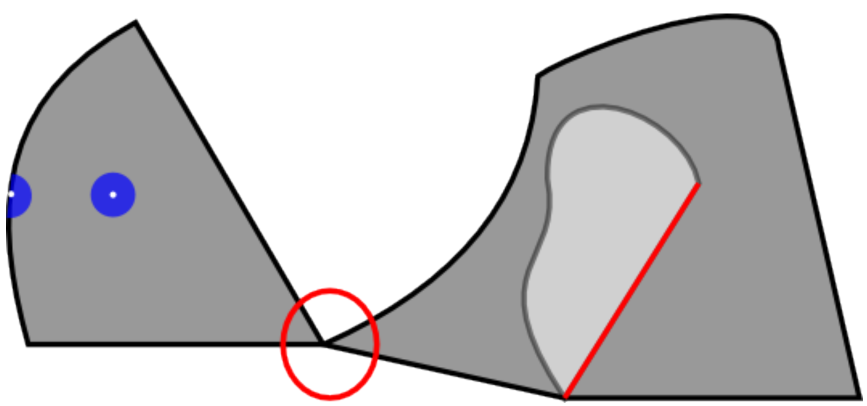
Topology vs geometry 2

Given a certain shape we can represent it in many different ways; topologically different but quite similar from a geometric point of view (demo klein bottle)

- Note that we can say many things on a given shape just by looking at its topology:
  - Manifoldness
  - Borders
  - Connected components
  - Orientability

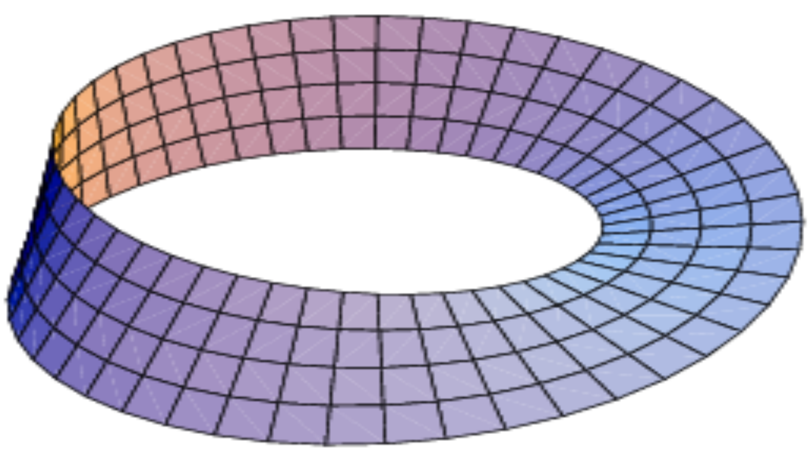
Manifoldness

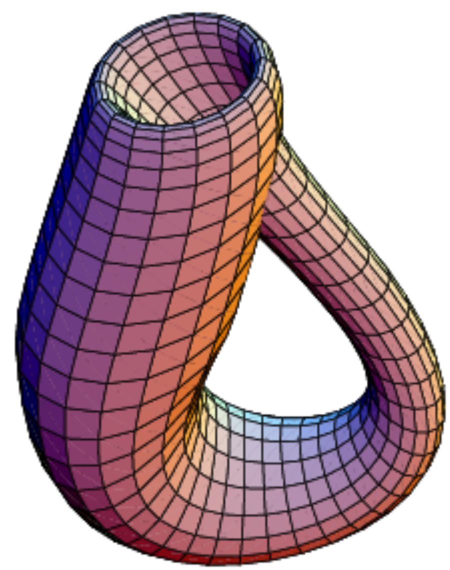
- a surface S is **2-manifold** *iff:*
  - the neighborhood of each point is homeomorphic to Euclidean space in two dimension
    or … in other words..
  - the neighborhood of each point is homeomorphic to a disk (or a semidisk if the surface has boundary)



Orientability

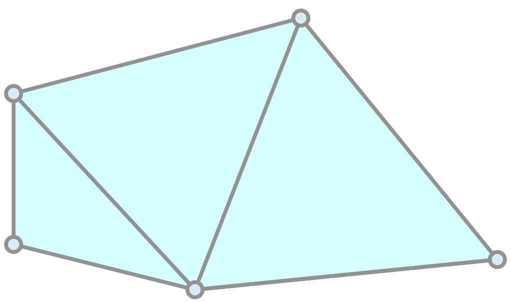
- A surface is **orientable** if it is possible to make a consistent choice for the normal vector
  - …it has two sides
- Moebius strips, klein bottles, and non manifold surfaces are not orientable





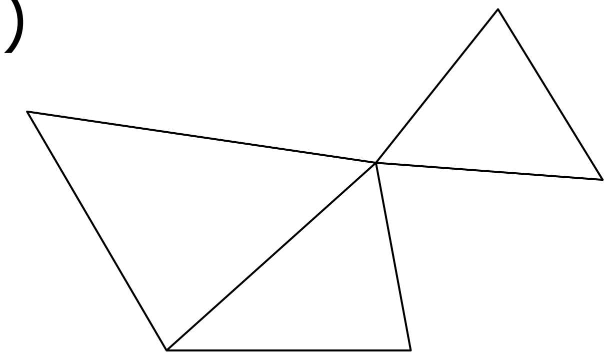
Adjacency/Incidency

- Two simplexes σ e σ' are **incident** if σ is a proper face of σ' (or viceversa)
- Two k-simplexes σ e σ' s are **m-adjacent** (k>m) if there exists a m-simplex that is a proper face of σ e σ'
  - Two triangles sharing an edge are 1-adjacent
  - Two triangles sharing a vertex are 0-adjacent



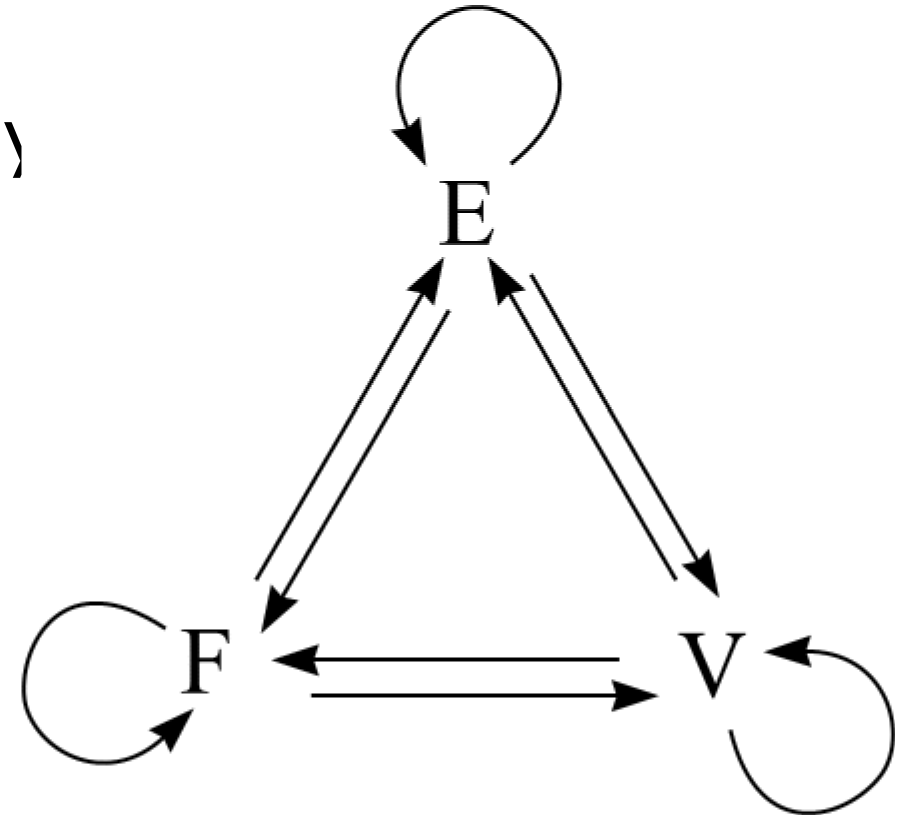
Adjacency Relations

- An intuitive convention to name practically useful topological relations is to use an *ordered* pair of letters denoting the involved entities:
  - **FF** edge adjacency between triangular **F**aces
  - **FV** from Faces to Vertices (e.g. the vertices composing a face)
  - **VF** from a vertex to a triangle (e.g. the triangles incident on a vertex)



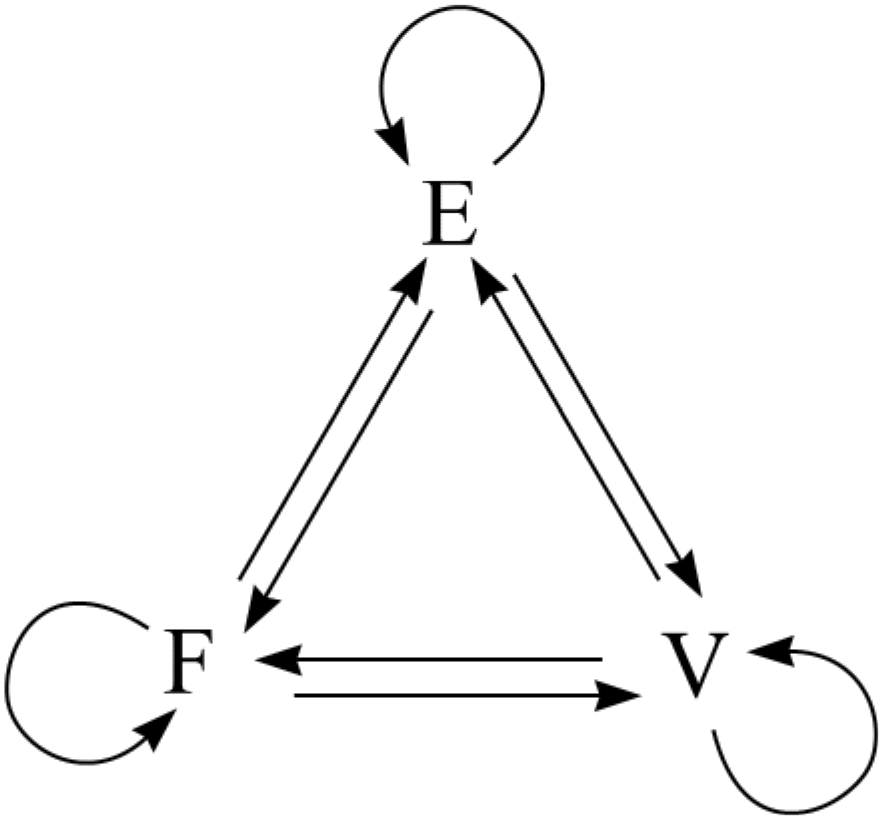
Adjacency Relationship

- Usually we only keep a small subset of all the possible adjacency relationships
- The other ones are procedurally generated



Adjacency Relation

- FF ~ 1-adjacency
- EE ~ 0 adjacency
- FE ~ proper subface of F with dim 1
- FV ~ proper subface of F con dim 0
- EV ~ proper subface of E con dim 0
- VF ~ F in Σ : V proper subface of F
- VE ~ E in Σ : V proper subface of E
- EF ~ F in Σ : E proper subface of F
- VV ~ V' in Σ : it exists an edge E:(V,V')



Partial adiacency

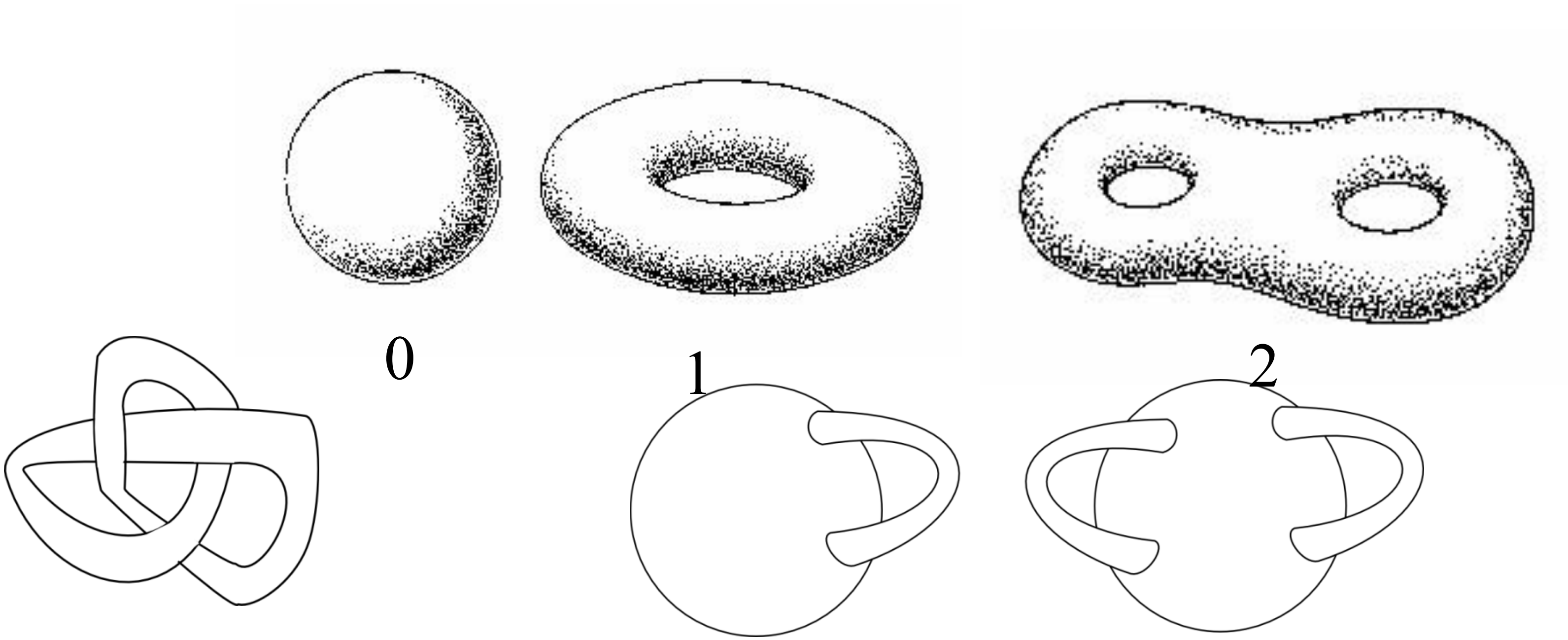
- For sake of conciseness it can be useful to keep only a partial information
  - VF* memorize only a reference from a vertex to a face and then surf over the surface using FF to find the other faces incident on V

Adjacency Relation

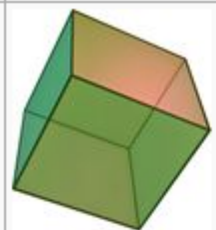
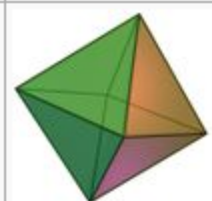
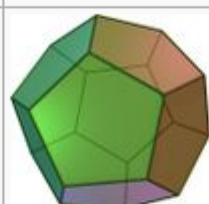
- For a two manifoldsimplicial 2-complex in R3
  - FV FE FF EF EV have bounded degree (are constant if there are no borders)
    - |FV|= 3 |EV| = 2 |FE| = 3
    - |FF| <= 2
    - |EF| <= 2
  - VV VE VF EE have variable degree but we have some avg. estimations:
    - |VV|~|VE|~|VF|~6
    - |EE|~10
    - F ~ 2V

Genus

- The **Genus** of a closed surface, orientable and 2-manifold is the maximum number of cuts we can make along non intersecting closed curves without splitting the surface in two.

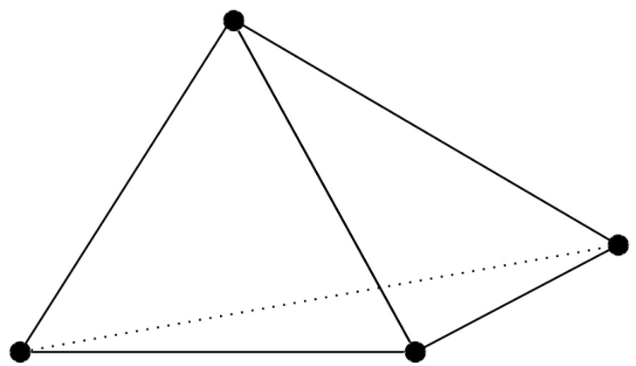


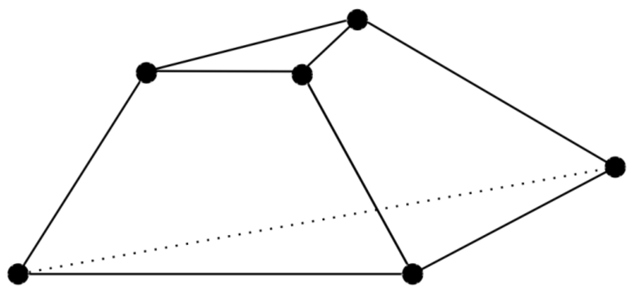
- ...also known as the number of *handles*

Name	Image	*V* (vertices)	*E* (edges)	*F* (faces)	Euler characteristic: *V* − *E* + *F*
Tetrahedron		4	6	4	**2**
Hexahedron or cube		8	12	6	**2**
Octahedron		6	12	8	**2**
Dodecahedron		20	30	12	**2**
Icosahedron		12	30	20	**2**

Euler characteristics

- χ = 2 for any *simply connected* polyhedron
- proof by construction…
- play with examples:



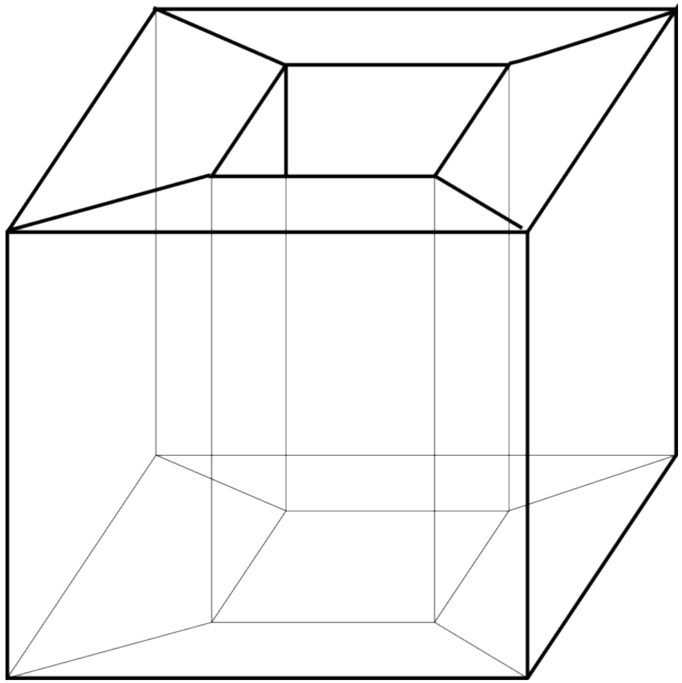


$$\chi = V - E + F$$
$$\chi = 4 - 6 + 4 = 2$$

$$\chi = (V + 2) - (E + 3) + (F + 1) =$$
$$\chi = (4 + 2) - (6 + 3) + (4 + 1) = 2$$

Euler characteristics

- let's try a more complex figure…



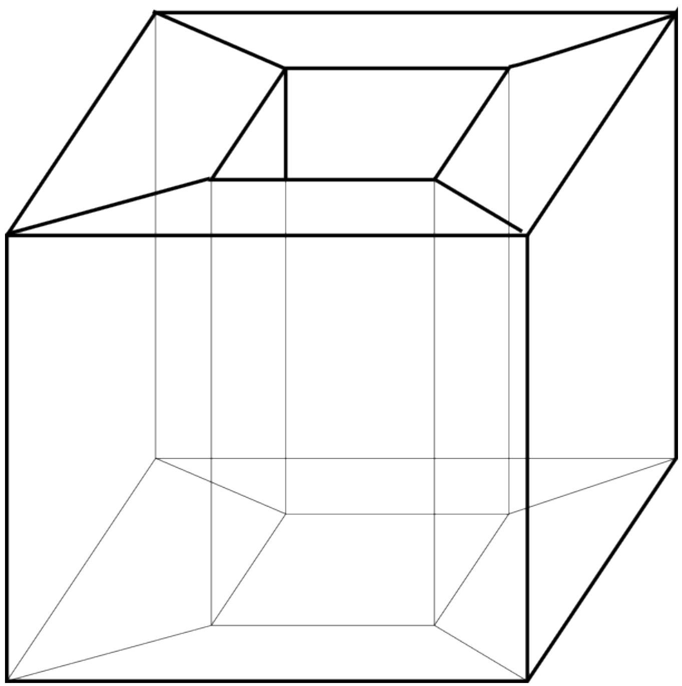
$$\chi = V - E + F$$
$$\chi = 16 - 32 + 16 = \mathbf{0}$$

- why =0 ?

Euler characteristics

$$\chi = 2 - 2g$$

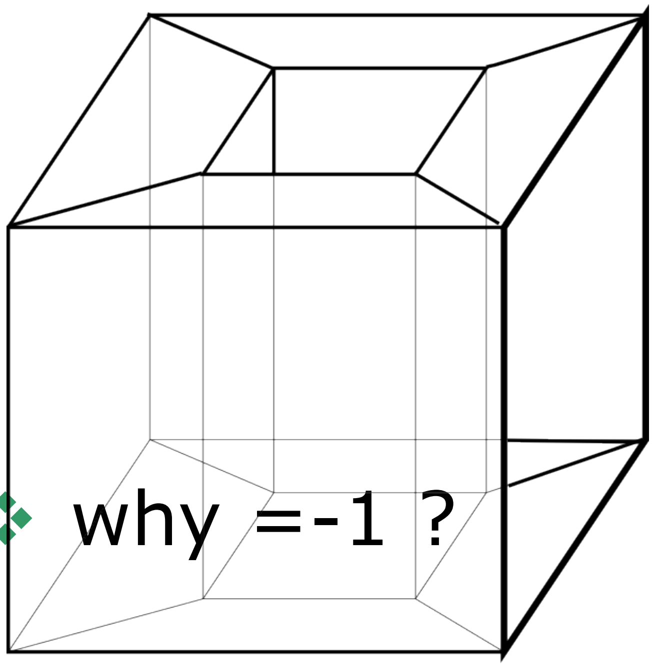
- where *g* is the genus of the surface



$$\chi = V - E + F$$
$$\chi = 16 - 32 + 16 = 0 = 2 - 2g$$

Euler characteristics

- let’s try a more complex figure…remove a face. The surface is not closed anymore



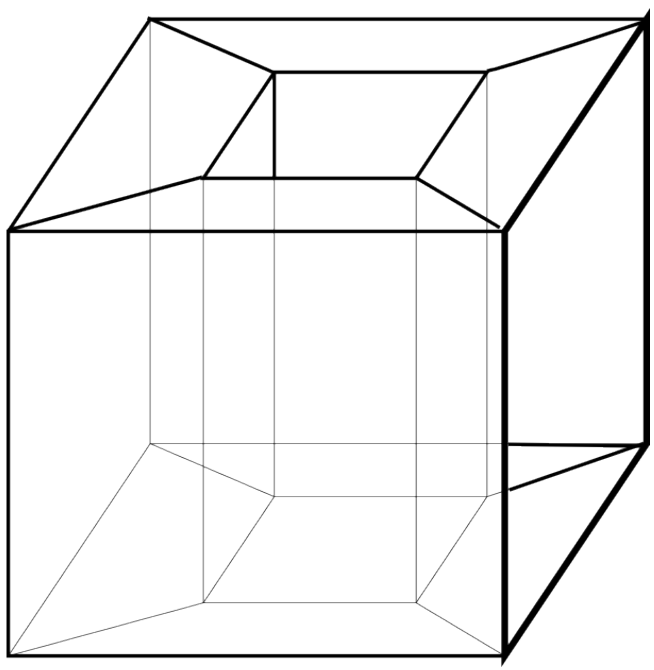
$$\chi = V - E + F$$
$$\chi = 16 - 32 + 15 = -\mathbf{1}$$

- why =-1 ?

Euler characteristics

$$\chi = 2 - 2g - b$$

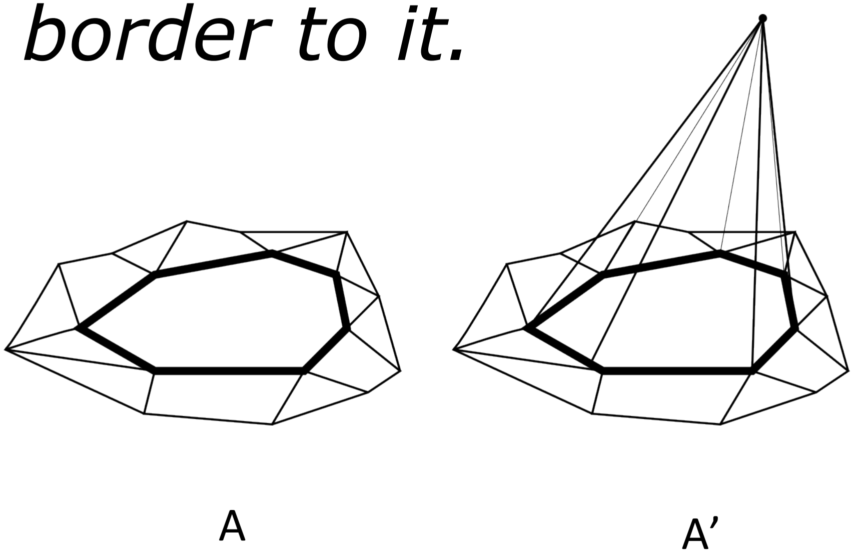
- where *b* is the number of borders of the surface



$$\chi = V - E + F$$
$$\chi = 16 - 32 + 15 = -1 = 2 - 2g - b$$

Euler characteristics

- *Remove the border by adding a new vertex and connecting all the **k** vertices on the border to it.*



$$X' = X + V' - E' + F' = X + 1 - k + k = X + 1$$

Differential quantities:normals

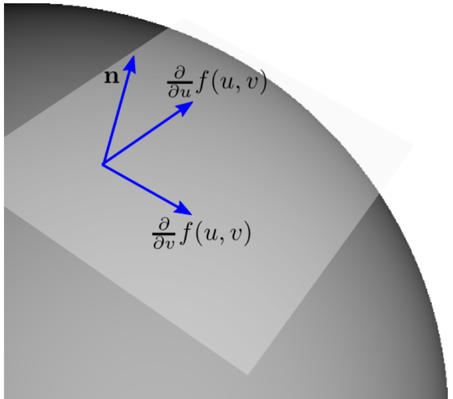
- The (unit) **normal** to a point is the (unit) vector perpendicular to the tangent plane

implicit surface: $f(x, y, z)=0$

$$n=\frac{\nabla f}{[\nabla f]}$$

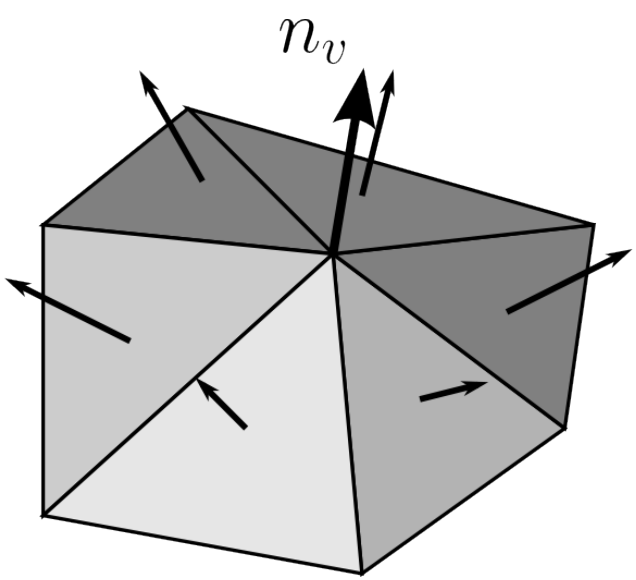
parametric surface:
$f(u,v)=(f_x(u,v), f_y(u,v), f_z(u,v))$

$$n=\frac{\partial}{\partial u}f\times\frac{\partial}{\partial v}f$$



Normals on triangle meshes

- Computed per-vertex and interpolated over the faces
- Common: consider the tangent plane as the average among the planes containing all the faces incident on the vertex

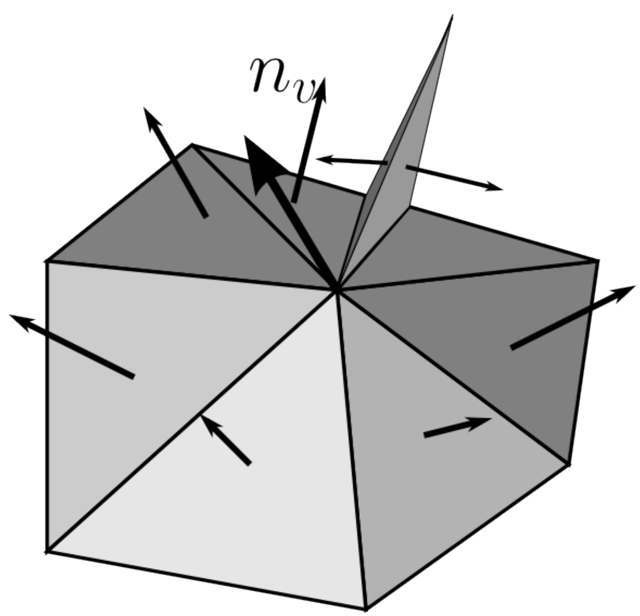


$$n_v = \frac{1}{\#N(v)} \sum_{f \in N(v)} n_f$$

$$N(v) = \{f : f \ coface \ of \ v\}$$

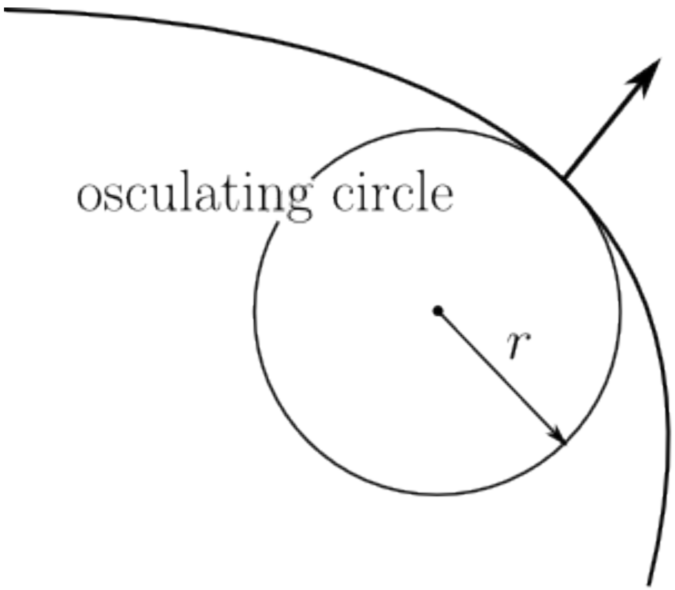
Normals on triangle meshes

- Does it work? Yes, for a “good” tessellation
  - Small triangles may change the result dramatically
  - Weighting by edge length / area / angle helps



Differential quantities: Curvature

- The curvature is a measure of how much a line is curve



r: radius or curvature
$\kappa = \frac{1}{r}$: curvature

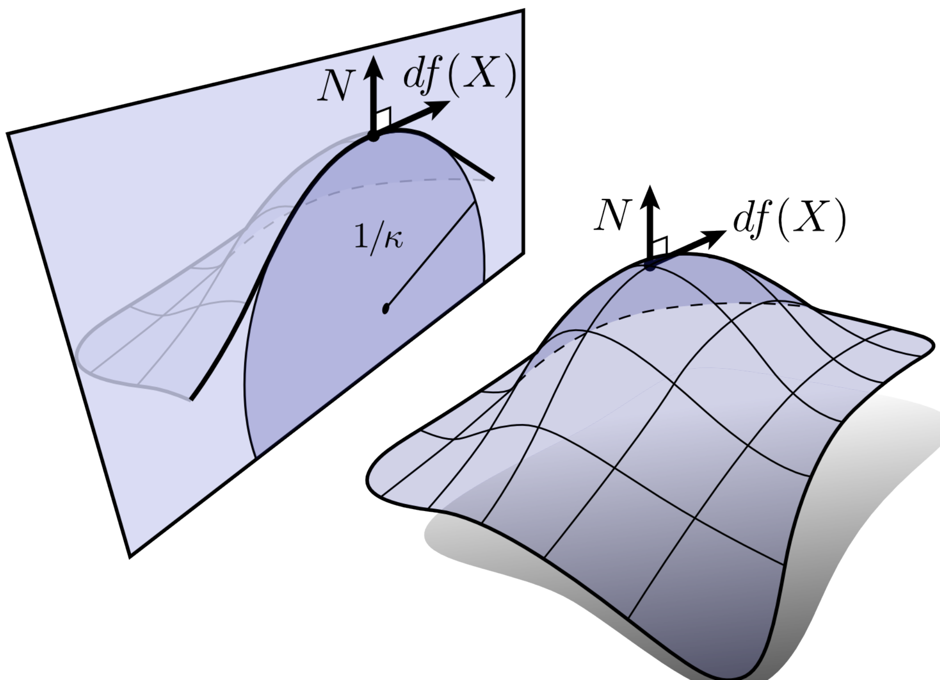
$x(t), y(t)$ *a parametric curve*
φ *tangential angle*
%s *arc lenght*

$$\kappa \equiv \frac{d\varphi}{ds} = \frac{d\varphi/dt}{ds/dt} = \frac{d\varphi/dt}{\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}} = \frac{d\varphi/dt}{\sqrt{x'^2 + y'^2}}$$

$$\kappa = \frac{x'y'' - y'x''}{(x'^2 + y'^2)^{3/2}}$$

Curvature on a surface

- Given the normal at point *p* and a tangent direction θ
- The curvature along θ is the 2D curvature of the intersection between the plane and the surface



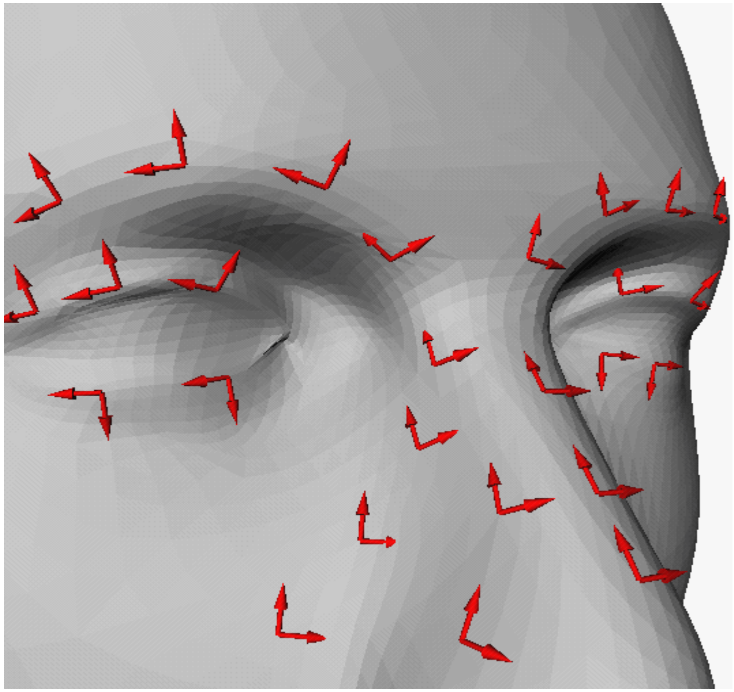
Curvatures

- A curvature for each direction
- Take the two directions for which curvature is max and min

κ_1, κ_2 *principal curvatures*

e_1, e_2 *principal directions*

- the directions of max and min curvature are orthogonal



[Meyer02]

Gaussian and Mean curvature

- Gaussian curvature: the product of principal curvatures

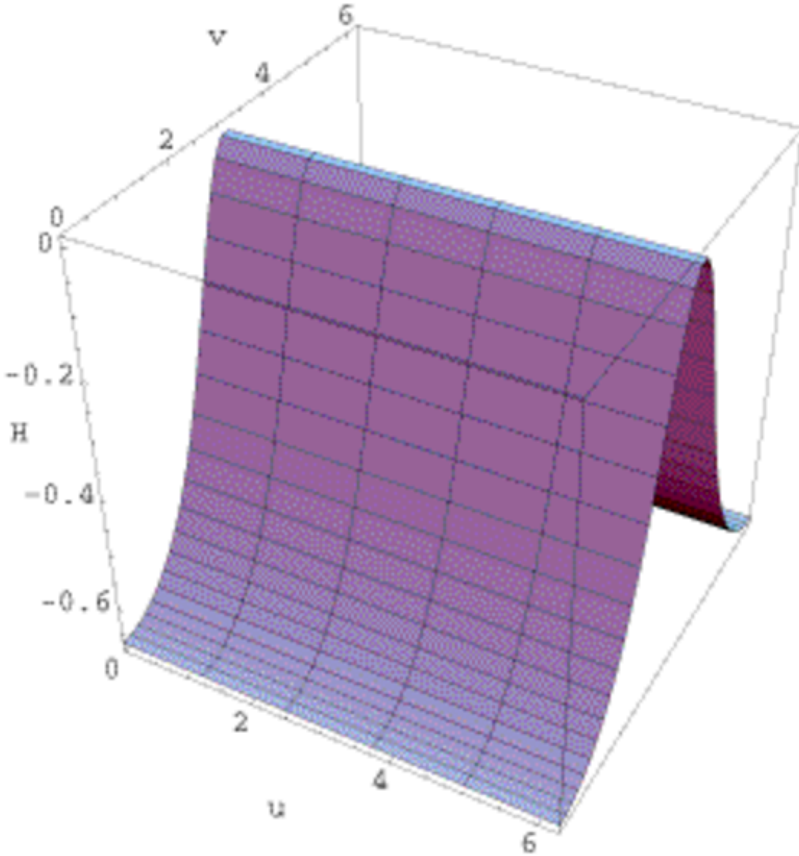
$$\kappa_G \equiv K \equiv \kappa_1 \cdot \kappa_2$$

- Mean curvature: the average of principal curvatures

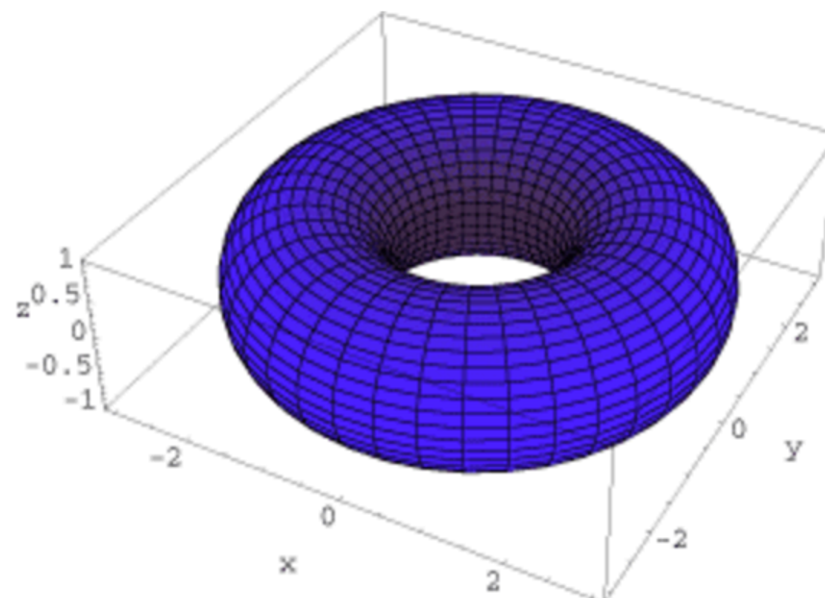
$$\bar{\kappa} \equiv H \equiv \frac{\kappa_1 + \kappa_2}{2}$$

Example a Torus

$$\begin{pmatrix} x[u, v] \\ y[u, v] \\ z[u, v] \end{pmatrix} = \begin{pmatrix} \mathrm{Cos}[u]\ (a + b\ \mathrm{Cos}[v]) \\ (a + b\ \mathrm{Cos}[v])\ \mathrm{Sin}[u] \\ b\ \mathrm{Sin}[v] \end{pmatrix}$$



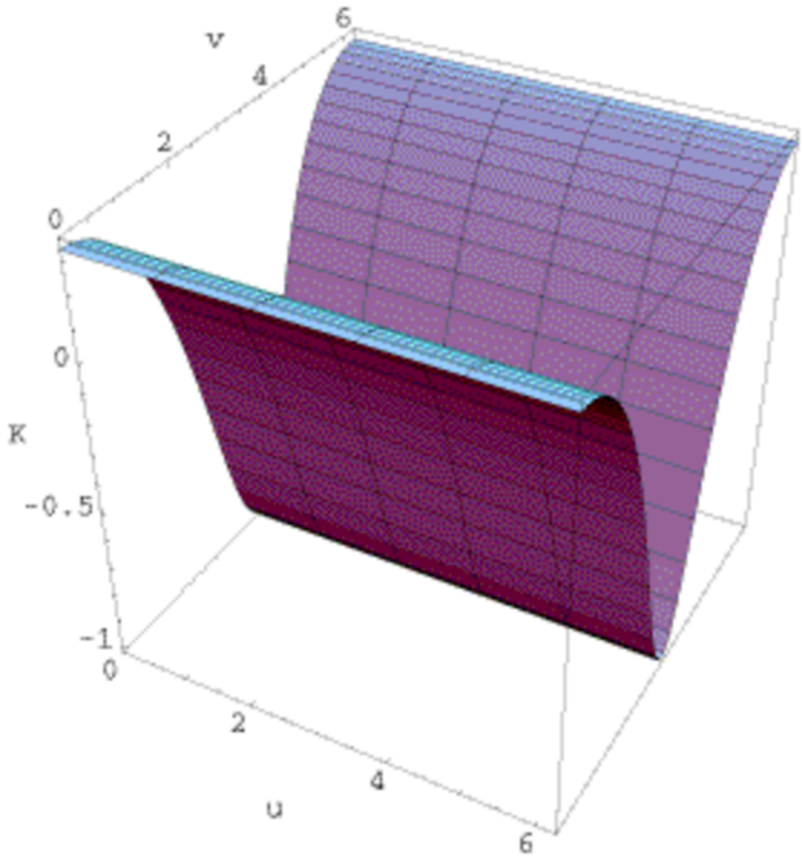
Mean curvature of the surface.



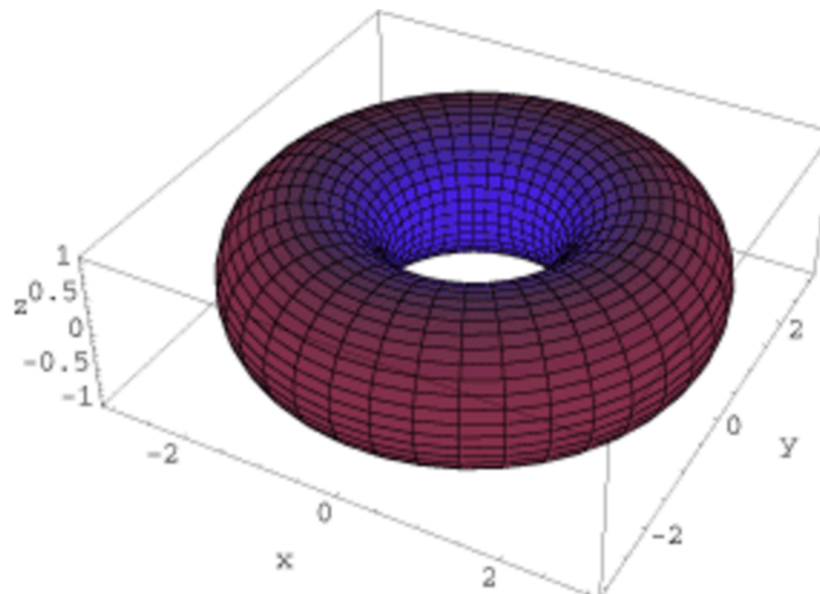
Surface colored by Mean curvature.

Example a Torus

$$\begin{pmatrix} x[u, v] \\ y[u, v] \\ z[u, v] \end{pmatrix} = \begin{pmatrix} \text{Cos}[u]\ (a + b\ \text{Cos}[v]) \\ (a + b\ \text{Cos}[v])\ \text{Sin}[u] \\ b\ \text{Sin}[v] \end{pmatrix}$$



Gaussian curvature of the surface.



Surface colored by Gaussian curvature.

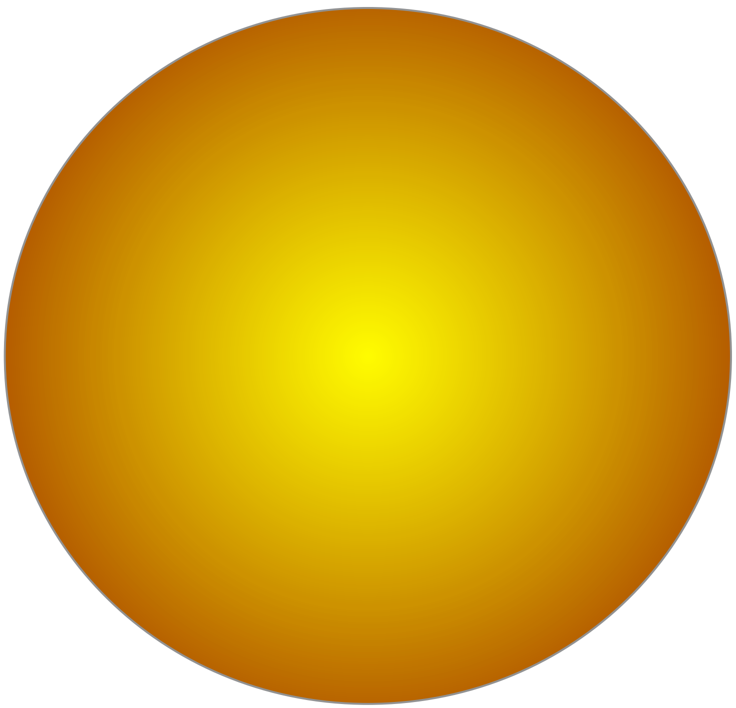
Gaussian Curvature

- Gaussian curvature is an intrinsic property
  - It can be computed by a bidimensional inhabitant of the surface by walking around a fixed point **p** and keeping at a distance *r*.

$$\kappa_G = \lim_{r \to 0} \left(2\pi r - C(r)\right) \cdot \frac{3}{\pi r^3}$$

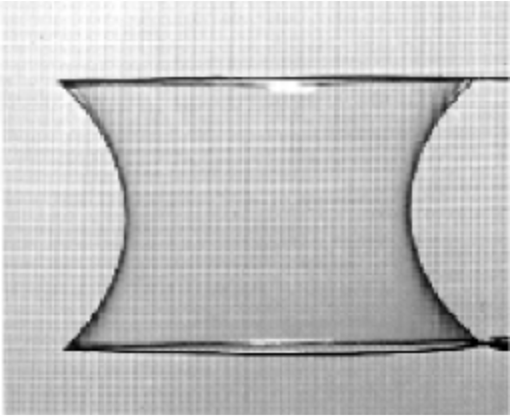
  with *C(r)* the distance walked
- Delevopable surfaces: surfaces whose Gaussian curvature is 0 everywhere

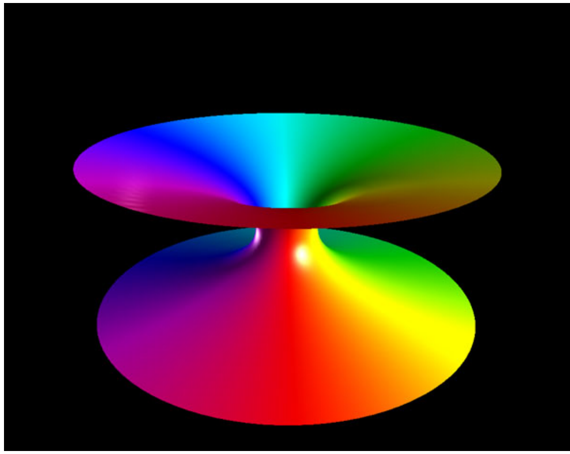
Gaussian Curvature



Mean Curvature

- Divergence of the surface normal
  - divergence is an operator that measures a vector field's tendency to originate from or converge upon a given point
- Minimal surface and minimal area surfaces
  - A surface is *minimal* when its mean curvature is 0 everywhere
  - All minimal area surfaces have mean curvature 0
- The surface tension of an interface, like a soap bubble, is proportional to its mean curvature





Mean Curvature

- Let A be the area of a disk around *p*. The mean curvature

$$2\bar{\kappa} n = \lim_{diam(A) \to 0} \frac{\nabla A}{A}$$

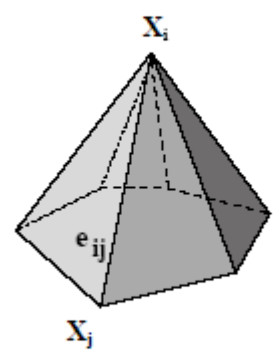
- the mean derivative is (twice) the divergence of the normal

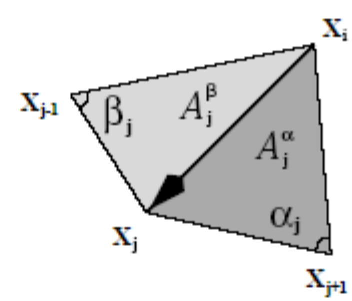
$$2\bar{\kappa} = \nabla \cdot n = \frac{\partial}{\partial x} n + \frac{\partial}{\partial y} n + \frac{\partial}{\partial z} n$$

Mean curvature on a triangle mesh

$$H(p)=\frac{1}{2A}\sum(\cot\alpha_i+\cot\beta_i)\,\|p-p_i\|$$

where α_j and β_j are the two angles opposite to the edge in the two triangles having the edge e_{ij} in common
A is the sum of the areas of the triangles





Gaussian curvature on a triangle mesh

- It's the *angle defect* over the area
- $$\kappa_G(v_i) = \frac{1}{3A}\left(2\pi - \sum_{t_j\, adj\, v_i} \theta_j\right)$$

- **Gauss-Bonnet Theorem:** The integral of the Gaussian Curvature on a closed surface depends on the Euler number

$$\int_S \kappa_G = 2\pi\chi$$

Mesh Data structures

- How to store geometry & connectivity?
  - compact storage
    - file formats
  - efficient algorithms on meshes
    - identify time-critical operations
    - all vertices/edges of a face
    - all incident vertices/edges/faces of a vertex

Mesh Data Structures

- how to store geometry & <u>connectivity</u>?
- compact storage
  - file formats
- efficient algorithms on meshes
  - identify time-critical operations
  - all vertices/edges of a face
  - all incident vertices/edges/faces of a vertex

Face Set (STL)

- face:
  - 3 positions

Triangles		
x_{11} y_{11} z_{11}	x_{12} y_{12} z_{12}	x_{13} y_{13} z_{13}
x_{21} y_{21} z_{21}	x_{22} y_{22} z_{22}	x_{23} y_{23} z_{23}
...	...	...
x_{F1} y_{F1} z_{F1}	x_{F2} y_{F2} z_{F2}	x_{F3} y_{F3} z_{F3}

36 B/f = 72 B/v

no connectivity!

Shared Vertex (OBJ, OFF)

- vertex:
  - position
- face:
  - vertex indices

Vertices
$x_1\ y_1\ z_1$
...
$x_V\ y_V\ z_V$

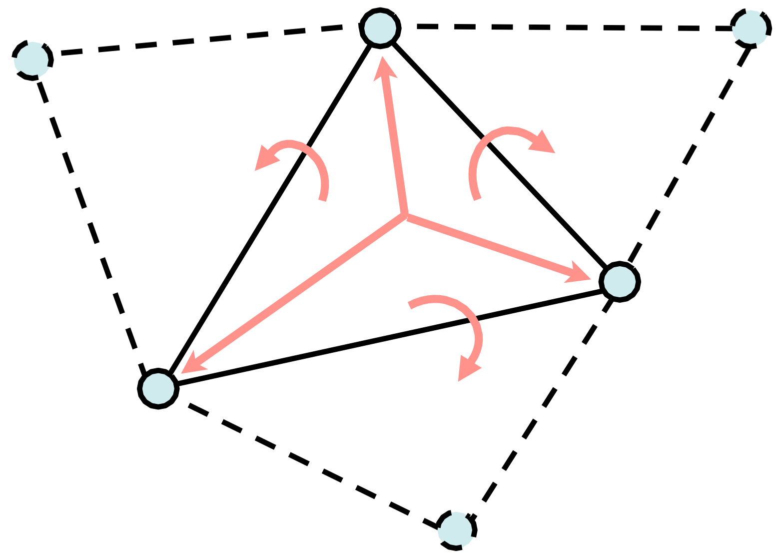
Triangles
$v_{11}\ v_{12}\ v_{13}$
...
...
...
...
$v_{F1}\ v_{F2}\ v_{F3}$

12 B/v + 12 B/f = 36 B/v

no neighborhood info

Face-Based Connectivity

- vertex:
  - position
  - 1 face
- face:
  - 3 vertices
  - 3 face neighbors

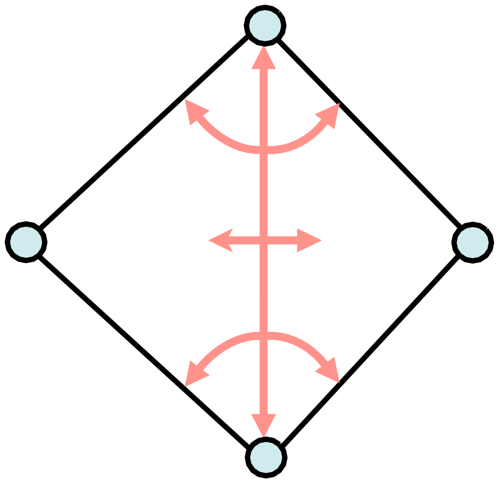


64 B/v

no edges!

Edge-Based Connectivity

- vertex
  - position
  - 1 edge
- edge
  - 2 vertices
  - 2 faces
  - 4 edges
- face
  - 1 edge

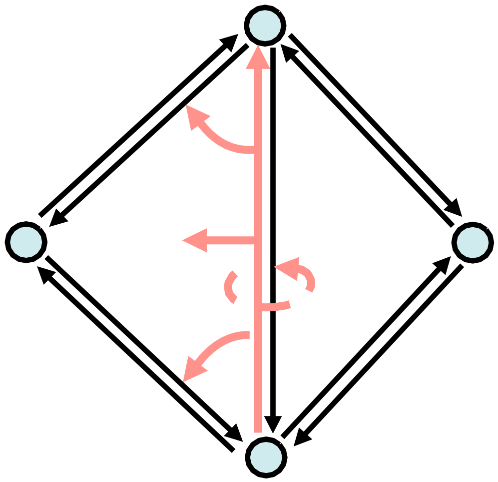


120 B/v

edge orientation?

Halfedge-Based Connectivity

- vertex
  - position
  - 1 halfedge
- halfedge
  - 1 vertex
  - 1 face
  - 1, 2, or 3 halfedges
- face
  - 1 halfedge



96 to 144 B/v

no case distinctions during traversal